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DYNAMICAL APPROACH FOR POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS IN UNBOUNDED DOMAINS

Dedicated to Klaus Kirchgässner on the occasion of his 70th birthday

We apply the trajectory dynamical systems approach to study the positive solutions of a semilinear elliptic problem in an unbounded domain Ω. The existence of the global attractor for the trajectory dynamical system associated with this problem is proved. The symmetrization and stabilization properties of positive solutions as $|x| \to \infty$ are also established in three dimensional case $\Omega \subset \mathbb{R}^3$.

1. Introduction.

It is well known that positive solutions of semilinear second order elliptic problems have symmetry and monotonicity properties which reflects the symmetry of the operator and of the domain, see e.g. [13, 6] for the case of bounded domains and [5, 7, 8, 9] for the case of unbounded domains (such as $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, cylindrical domains, etc.).

These results have been extended to the case of positive solutions of second order parabolic problems in bounded symmetric domains in [16, 3, 4]. Moreover, the symmetrization and stabilization properties of such solutions as $t \to \infty$ were investigated using the combination of moving planes method with the classical methods of dynamical systems theory (such as ω-limit sets, attractors, etc.).

The main goal of the present paper is to apply the dynamical approach to study the symmetrization and stabilization as $|x| \to \infty$ properties of positive solutions of elliptic problems in asymptotically symmetric unbounded domains. To the best of our knowledge the use of dynamical systems methods for elliptic problems was initiated in the pioneering paper of K.Kirchgässner [18] where a local center manifold for a semilinear elliptic equation on a strip was constructed, see also [10, 20, 17, 14] for further development and applications of this construction.

One of the main difficulties which arises in dynamical study of elliptic equations is the fact that the corresponding Cauchy problem is not well posed for such equations, and consequently the straightforward interpretation of the elliptic equation as an evolution equation leads to semigroups of multivalued maps even in the case of cylindrical domains, see [2]. The usage of multivalued maps can be overcome using the so-called trajectory dynamical approach (see [25, 27, 22]). Under this approach one fixes a signed direction $\vec{l}$ in $\mathbb{R}^n$ which will play the role of time. Then the space K^+ of all bounded solutions of the elliptic problem in the unbounded domain Ω is considered as a trajectory phase space for the semi-flow $T_h^{\vec{l}}$ of translations along the direction $\vec{l}$ defined via

$$(T_h^{\vec{l}}u)(x) := u(x + h\vec{l}), \ h \in \mathbb{R}_+, \ u \in K^+$$

In order the trajectory dynamical system $(T_h^{\vec{l}}, K^+)$ to be well defined one evidently needs the domain Ω to be invariant with respect to positive translations along the $\vec{l}$ directions:

$$T_h^{\vec{l}}\Omega \subset \Omega, \ \ T_h^{\vec{l}}x := x - h\vec{l}$$

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The above approach was applied in [25, 22, 21] for study the elliptic boundary value problems in cylindrical domains and in [27] for more general class of unbounded domains, see also [23, 11] for application to evolution problems for which the uniqueness problem is not solved yet (e.g., for 3D Navier-Stokes equations) and [12] for another possibility to avoid the usage of multivalued maps in the case of elliptic equations in cylindrical domains.

In this paper we apply the trajectory dynamical systems approach to more detailed study the asymptotic behavior of positive solutions of the following model elliptic boundary problem in an unbounded domain $x := (x_1, x_2, x_3) \in \Omega_+ := \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$:

$$\begin{cases} \Delta_x u - f(u) = 0; \\ u\big|_{x_1=0} = u_0; \quad u\big|_{x_2=0} = 0; \end{cases} \tag{1.1}$$

It is assumed that the nonlinear term $f(u)$ satisfies the following conditions:

$$\begin{cases} 1.\ f \in C^1(\mathbb{R}, \mathbb{R}), \\ 2.\ f(v).v \geq -C + \alpha|v|^2, \quad \alpha > 0, \\ 3.\ f(0) \leq 0 \end{cases} \tag{1.2}$$

(see the last section for the relaxation of these conditions).

As it is mentioned above we consider nonnegative solutions of problem (1.1):

$$u(x) \geq 0 \ x \in \Omega_+ \tag{1.3}$$

and study their behavior when $x_1 \to +\infty$. Thus, in our situation the x_1-axis will play the role of time ($\vec{l} := (1, 0, 0)$). Moreover we restrict ourselves to consider only bounded with respect to $x \to \infty$ solutions of (1.1). To be more precise, a bounded solution of (1.1) is understood to be a function $u \in C_b^{2+\beta}(\overline{\Omega_+})$ for some fixed $0 < \beta < 1$, which satisfies (1.1) in a classical sense (in a fact due to the interior estimates this assumption is equivalent to $u \in C_b(\overline{\Omega_+})$ but we prefer to work with classical solutions). Therefore the boundary data is assumed to be nonnegative $u_0(x_2, x_3) \geq 0$ and belonging to the space

$$u_0 \in C_b^{2+\beta}(\Omega_0), \quad (x_2, x_3) \in \Omega_0 := \mathbb{R}_+ \times \mathbb{R}^n \tag{1.4}$$

Here and below we denote

$$C_b^{2+\beta}(V) := \{u_0 : \|u_0\|_{C_b^{2+\beta}} := \sup_{\xi \in V} \|u_0\|_{C^{2+\beta}(B_\xi^1 \cap V)} < \infty\} \tag{1.5}$$

where B_ξ^r means a ball of radius r centered in ξ.

The paper is organized as follows.

The 'dissipative' with respect to $x_1 \to \infty$ a priori estimate for the positive solutions of (1.1) which allows to apply the trajectory approach to our situation and in particular gives the existence of at least one nonnegative solution of (1.1) is derived in Section 2.

In Section 3 we construct the trajectory dynamical system (T_h, K^+) associated with problem (1.1), where $K^+ \subset C^{2+\beta}(\overline{\Omega_+})$ is a set of all bounded nonnegative solutions of problem (1.1) (with all admissible boundary data u_0) endowed by the local topology induced by the embedding

$$K^+ \subset C_{loc}^{2+\beta}(\overline{\Omega})$$

and $(T_h u)(x_1, x_2, x_3) := u(x_1 + h, x_2, x_3)$ is the translation with respect to x_1 direction. Moreover, it is proved here that the dynamical system thus obtained possesses a global attractor $\mathcal{A}_{tr}$ which is called the trajectory attractor of equation (1.1).

Section 4 is devoted to a more comprehensive study of three dimensional case ($n = 1$, $\Omega_+ = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$). In this case, using the symmetry result for bounded solutions of problem (1.1) in the *half-space* $\Omega := \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ obtained in [9] we establish that the attractor $\mathcal{A}_{tr}$ consists of functions $u(x) := V(x_2)$ which are independent of x_1 and x_3 and consequently satisfy the ordinary equation

$$V''(x_2) - f(V(x_2)) = 0, \quad V(0) = 0, \quad V \geq 0 \tag{1.6}$$

Note that the set of bounded positive solutions of ODE (1.6) can be completely described using e.g. the standard phase portrait technique. Using this description we prove finally that every positive solution $u(x)$ of problem (1.1) stabilizes as $x_1 \to \infty$ to one of the positive bounded solution $V = V_u(x_2)$ of problem (1.6).

A number of remarks which explains the imposed restrictions to the nonlinearity is given in Section 5.

2. A priori estimates and solvability results.

In this Section we prove that problem (1.1) possesses at least one non-negative bounded solution u and derive the estimate for such solutions which is of fundamental significance in order to apply the dynamical approach to elliptic equation (1.1).

The main result of this Section is the following theorem.

THEOREM 2.1. *Let $u_0 \in C_b^{2+\beta}(\Omega_0)$ and let the first and second compatibility conditions at $\partial\Omega_0$ are valid (i.e. $u_0(0, x_3) = 0$ and $\partial^2_{x_2} u_0(0, x_3) = f(0)$). Then* (1.1) *possesses at least one nonnegative bounded solution and every such solution u satisfies the estimate*

$$\|u\|_{C^{2+\beta}(B^1_x \cap \Omega_+)} \leq Q(\|u_0\|_{C_b^{2+\beta}}) e^{-\gamma x_1} + C_f \tag{2.1}$$

$x = (x_1, x_2, x_3) \in \Omega_+$, $\gamma > 0$, *Q is an appropriate monotonic function, and C_f is independent of u_0.*

Proof. Let us firstly verify a priori estimate (2.1). To this end we consider (as usual) the function $w(t, x) = u^2(t, x)$ which evidently satisfies the equation

$$\Delta_x w = 2f(u).u + 2\nabla_x u.\nabla_x u \geq -2C + 2\alpha w, \quad w\big|_{x_1=0} = u_0^2, \quad w\big|_{x_2=0} = 0 \tag{2.2}$$

Consider also the auxiliary linear problem

$$\Delta_x w_1 = -2C + 2\alpha w_1, \quad w_1\big|_{x_1=0} = w\big|_{x_1=0} = u_0^2, \quad w_1\big|_{x_2=0} = 0 \tag{2.3}$$

with the same boundary conditions as for the function w.

LEMMA 2.1. *Linear equation* (2.3) *possesses a unique bounded solution $w_1(x)$ which satisfies the following estimate:*

$$\|w_1\|_{C(B^1_x)} \leq C_1 \|u_0\|^2_{C_b(\Omega_0)} e^{-\alpha x_1} + C_2 \tag{2.4}$$

Proof. The proof of the lemma is standard and based on the maximum principle. Indeed, let us decompose $w_1(x) = v(x) + v_1(x)$, where $v_1(x)$ is a solution of non-homogeneous equation

(2.3) with zero boundary conditions and $v(x)$ is a solution of homogeneous equation with non-zero boundary conditions. Then, evidently,

$$\|v_1\|_{C_b(\Omega_+)} \le C_2 \tag{2.5}$$

with C_2 depends only on C in the right-hand side of (2.3). In order to obtain the exponential decaying of $v(x)$ we introduce the functions $\psi_h(x) := 1/\cosh(\varepsilon(x_1 - h))$ where $\varepsilon > 0$ is a sufficiently small number and $h \ge 0$. Then it is not difficult to verify that

$$|\psi_h'(x)| \le \varepsilon\psi_h(x), \quad |\psi_h''(x)| \le 3\varepsilon^2\psi_h(x), \quad \psi_h(0) \le e^{-\varepsilon h} \tag{2.6}$$

Let $v_h(x) := \psi_h(x)v(x)$, then this function satisfies the equation

$$\Delta_x v_h + L_h^1(x)v_h + L_h^2(x)\partial_{x_1}v_h - 2\alpha v_h = 0, \quad v_h\big|_{x_1=0} = \psi_h(0)u_0^2 \tag{2.7}$$

and (2.6) implies that $|L_h^1(x)| \le C\varepsilon^2$ and $|L_h^2(x)| \le C\varepsilon$ (where C is independent of h). Consequently if $\varepsilon > 0$ is small enough the classical maximum principle works for equation (2.7), therefore

$$\|v_h\|_{C_b(\Omega_+)} \le \|v_h\|_{C_b(\Omega_0)} \tag{2.8}$$

Estimate (2.4) is an immediate corollary of (2.5) (2.8) and the third estimate of (2.6). Lemma 1.1 is proved.

Having estimate (2.4) and applying the comparison principle to the solutions w and w_1 of (2.2) and (2.3) respectively and the evident fact that $w = u^2$ is non-negative we derive that

$$\|u\|^2_{C(B_x^1)} \le \|w\|_{C(B_x^1)} \le \|w_1\|_{C(B_x^1} \le C_1\|u_0\|_{C_b^{2+\beta}(\Omega_0)}e^{-\alpha x_1} + C_2 \tag{2.9}$$

Recall, that due to classical interior estimates for the Laplace equation (see e.g. [19]), we have the following estimate for every small positive $\delta > 0$

$$\begin{aligned}\|u\|_{C^{2-\delta}(\Omega_+\cap B_x^1)} &\le \\ &\le C(\|f(u)\|_{L^\infty(\Omega\cap B_x^1)} + \|u\|_{L^\infty(\Omega\cap B_x^2)} + \chi(2-x_1)\|u_0\|_{C^{2+\beta}(\Omega_0\cap B_x^2)}) \le \\ &\le Q(\|u\|_{L^\infty(\Omega_+\cap B_x^2)}) + C\chi(2-x_1)\|u_0\|_{C^{2-\delta}(\Omega_0\cap B_x^2)}, \quad x \in \Omega_+\end{aligned} \tag{2.10}$$

where the monotonic function Q and the constant C depend only on f and α and independent of $x \in \Omega$ and of the concrete solution u, and $\chi(z)$ is a classical Heaviside function (which equals zero for $z \le 0$ and one for $z > 0$). Remind that we assume the first compatibility condition $u_0\big|_{x_2=0} = 0$ to be valid. This assumption is necessary in order to obtain $C^{2-\delta}$-regularity in (2.10) in the case where x is near to the edge $\partial\Omega_0$).

Inserting now estimate (2.9) into the right-hand side of (2.10) we derive the analogue of estimate (2.1) for $C^{2-\delta}$-norm:

$$\|u\|_{C^{2-\delta}(B_x^1\cap\Omega_+)} \le Q(\|u_0\|_{C_b^{2-\delta}})e^{-\gamma x_1} + C_f \tag{2.10'}$$

In order to derive estimate (2.1) it is sufficient to use now the elliptic interior estimate in the form

$$\begin{aligned}\|u\|_{C^{2+\beta}(\Omega_+\cap B_x^1)} &\le \\ &\le C(\|f(u)\|_{C^1(\Omega\cap B_x^1)} + \|u\|_{C(\Omega\cap B_x^2)} + \chi(2-x_1)\|u_0\|_{C^{2+\beta}(\Omega_0\cap B_x^2)}) \le \\ &\le Q(\|u\|_{C^1(\Omega_+\cap B_x^2)}) + C\chi(2-x_1)\|u_0\|_{C^{2+\beta}(\Omega_0\cap B_x^2)}, \quad x \in \Omega_+\end{aligned}$$

(here we have implicitly used the second compatibility condition $\partial^2_{x_2} u\big|_{x_2=0} = f(0)$ in order to obtain $C^{2+\beta}$ regularity near the edge $\partial\Omega_0$). Inserting estimate (2.10′) into the last interior estimate we derive inequality (2.1) for the $C^{2+\beta}$-norm.

Let us verify now the existence of a *positive* solution for problem (1.1). To this end we consider a sequence of bounded domains Ω_+^N, $N \in \mathbb{N}$, defined via

$$\Omega_+^N := \Omega_+ \cap B_0^{N+1}$$

and a sequence of cut-off functions $\phi_N(x) \equiv 1$ if $x \in B_0^N$ and $\phi_N(x) \equiv 0$ if $x \notin B_0^{N+1}$, $0 \le \phi \le 1$. Consider also the family of auxiliary elliptic problems

$$\Delta_x u^N - f(u^N) = 0,\ x \in \Omega_+^N,\ \ u^N\big|_{\partial\Omega_+^N \cap \Omega_0} = u_0\phi_N,\ \ u^N\big|_{\partial\Omega_+^N \setminus \Omega_0} = 0 \qquad (2.11)$$

Note that, according to our construction, $u^N\big|_{\partial\Omega_+^N} \ge 0$ and, according to assumptions (1.2), $w_-^N \equiv 0$ is a subsolution and $w_+^N \equiv R$ is a supersolution for (2.11) if R is large enough. Thus (see e.g. [26]), problem (2.11) has at least one $non-negative$ solution $R \ge u^N(x) \ge 0$. Note that R is in a fact independent of N. Consequently, applying again the interior regularity theorem (see estimate (2.10)) we derive that

$$\|u^N\|_{C^{2+\beta}(B_x^1 \cap \Omega_+^N)} \le C \qquad (2.12)$$

with $C = C(f, u_0)$ is independent of N and $x \in \Omega_+^N$.

Having uniform estimate (2.12) one can easily pass to the limit $N \to \infty$ in equation (2.11) and construct a bounded $non-negative$ solution $u(x)$ of initial equation (1.1). Theorem 2.1 is proved.

3. The attractor.

In this Section we study the behavior of the non-negative solutions of problem (1.1) when $x_1 \to \infty$ applying the dynamical system approach to elliptic boundary value problem (1.1) in the unbounded domain Ω_+.

Recall that under such consideration we should fix some direction in our unbounded domain Ω_+ and interpret it as the 'time' direction (see [27]). In our case it will be the x_1-direction then x_1 variable will play the role of 'time' variable and we (formally) will consider (1.1) as an 'evolutionary' equation in an unbounded domain Ω_0. The main difficulty which arises here is the fact that the solution of (1.1) may be not unique and consequently we cannot construct the semigroup corresponding to 'evolutionary' equation (1.1) in the ordinary way.

One of possible ways to overcome this difficulty is to use the trajectory approach which takes into the accordance the dynamical system to (1.1) in another way (following to [25, 27]). Namely, let us consider the union K^+ of all bounded positive solutions of (1.1) which corresponds to every $u_0 \in C_b^{2+\beta}$. Then a semigroup of positive shifts

$$(T_h u)(x_1, x_2, x_3) := u(x_1 + h, x_2, x_3) \qquad (3.1)$$

acts on the set K^+:

$$T_h : K^+ \to K^+,\ \ K^+ \subset C_b^{2+\beta}(\overline{\Omega_+}) \qquad (3.2)$$

This semigroup acting on K^+ is called the trajectory dynamical system, corresponding to (1.1). Our next task is to construct the attractor for this system. Firstly we note that the uniform topology of $C_b^{2+\beta}$ is too strong for our purposes. That is why we endow the space K^+ by a local topology according to the embedding

$$K^+ \subset C_{loc}^{2+\beta}(\overline{\Omega_+}) \tag{3.3}$$

where by definition $\Phi := C_{loc}^{2+\beta}(\overline{\Omega_+})$ is a Frechet space generated by seminorms $\|\cdot\|_{C^{2+\beta}(B^1_{x_0}\cap\Omega_+)}$, $x_0 \in \Omega_+$.

Recall briefly the definition of the attractor adopted to our case.

DEFINITION 3.1 *The set $\mathcal{A}_{tr} \subset K^+$ is called the attractor for trajectory dynamical system* (3.2) *(= trajectory attractor for problem* (1.1)*) if the following conditions are valid.*

1. The set $\mathcal{A}_{tr}$ is compact in $C_{loc}^{2+\beta}(\overline{\Omega_+})$.

2. It is strictly invariant with respect to T_h: $T_h\mathcal{A}_{tr} = \mathcal{A}_{tr}$

3. $\mathcal{A}_{tr}$ attracts bounded subsets of solutions when $x_1 \to \infty$. It means that for every bounded (in the uniform topology of $C_b^{2+\beta}$) subset $B \subset K^+$ and for every neighborhood $\mathcal{O}(\mathcal{A}_{tr})$ in $C_{loc}^{2+\beta}$ topology there exists $H = H(B, \mathcal{O})$ such that

$$T_h B \subset \mathcal{O}(\mathcal{A}_{tr}) \text{ if } h \geq H \tag{3.4}$$

Note that the first assumption of the definition claims that the restriction $\mathcal{A}_{tr}\big|_{\Omega_1}$ is compact in $C^{2+\beta}(\overline{\Omega_1})$ for every bounded $\Omega_1 \subset \Omega_+$ and the third one is equivalent to the following:

For every bounded subdomain $\Omega_1 \subset \Omega_+$, for every B – bounded subset of K^+ and for every neighborhood $\mathcal{O}(\mathcal{A}_{tr}\big|_{\Omega_1})$ in $C^{2+\beta}(\overline{\Omega_1})$-topology of the restriction $\mathcal{A}_{tr}$ to this domain there exists $H = H(\Omega_1, B, \mathcal{O})$ such that

$$(T_h B)\big|_{\Omega_1} \subset \mathcal{O}(\mathcal{A}_{tr}\big|_{\Omega_1}) \text{ if } h \geq H \tag{3.5}$$

THEOREM 3.1.*Let the assumptions of Theorem 2.1 hold. Then equation* (1.1) *possesses the trajectory attractor $\mathcal{A}_{tr}$ which has the following structure:*

$$\mathcal{A}_{tr} = \Pi_{\Omega_+} K(\Omega) \tag{3.6}$$

where $(x_1, x_2, x_3) \in \Omega := R \times \mathbb{R}_+ \times \mathbb{R}^n$ and symbol $K(\Omega)$ means the union of all bounded nonnegative solutions $\widehat{u}(x) \in C_b^{2+\beta}(\Omega)$ of

$$\Delta_x \widehat{u} - f(\widehat{u}) = 0, \quad x \in \Omega, \quad u\big|_{\partial\Omega} = 0, \quad \widehat{u}(x) \geq 0 \tag{3.7}$$

I.e. the attractor $\mathcal{A}_{tr}$ consists of all bounded nonnegative solutions u of (1.1) *in Ω_+ which can be extended to bounded nonnegative solution $\widehat{u}$ in Ω.*

Proof. As usual (see e.g. [1]) in order to verify that a semigroup $T_h : K^+ \to K^+$ possesses an attractor we should verify that this semigroup is continuous for every fixed $h \geq 0$ and that this semigroup possesses a compact attracting (or absorbing) set in K^+.

The continuity of the semigroup T_h on K^+ is obvious in our situation. Indeed, the semigroup T_h of positive shifts along the x_1 axis is evidently continuous (for every fixed h) as a semigroup in $C_{loc}^{2+\beta}(\overline{\Omega_+})$ therefore it's restriction to K^+ is also continuous.

Thus it remains to construct a compact absorbing set for $T_h : K^+ \to K^+$.

Let $\mathbb{B}_R$ be the R-ball centered in 0 in the space $C_b^{2+\beta}(\Omega_+)$. Then estimate (1.1) implies that the set

$$\mathbb{M}_R := K^+ \cap \mathbb{B}_R$$

will be an absorbing set for semigroup (3.2) (more precisely for $R = 2C_f$ where C_f is defined in (1.1)). But this set is not compact in Φ. That is why we construct a new set

$$\mathbb{V}_R := T_1\mathbb{M}_R \subset \mathbb{M}_R \subset K^+ \tag{3.8}$$

Evidently this set is also absorbing. We claim also that this set is precompact in Φ. Indeed, by definition the set $\mathbb{V}_R$ consists of all bounded solutions u of equation (1.1) which can be extended to bounded solution $\widehat{u}$, defined not for $x_1 \geq 0$, but for $x_1 \geq -1$, such that

$$\|\widehat{u}\|_{C_b^{2+\beta}([-1,\infty]\times\Omega_0)} \leq R \tag{3.9}$$

Note now that, due to (1.2), $f \in C^1$ consequently we may apply the interior estimate (see (2.10)) for the solution $\widehat{u}$ not only with the exponent $2+\beta$, but with an arbitrary one $2+\beta'$ with $\beta' < 1$. Particularly, if we fix $\beta' > \beta$ then the interior estimate together with (3.9) gives us that

$$\|u\|_{C_b^{2+\beta'}(T_1\Omega_+)} = \|\widehat{u}\|_{C_b^{2+\beta'}(\Omega_+)} \leq R_1 \tag{3.10}$$

where the constant R_1 depends only on R and f. Consequently, we have proved that the set

$$\mathbb{V}_R \subset C_b^{2+\beta'}(\Omega_+) \tag{3.11}$$

and is bounded in it. Note now that the embedding $C_b^{2+\beta'}(\Omega_+) \subset \Phi$ is compact if $\beta' > \beta$ and consequently $\mathbb{V}_R$ is really precompact in Φ. (This was the main reason to endow the trajectory phase space by the 'local' topology of Φ but not by the 'uniform' topology of $C_b^{2+\beta}(\Omega_+)$. Indeed, the embedding $C_b^{2+\beta'}(\Omega_+) \subset C_b^{2+\beta}(\Omega_+)$ is evidently non-compact and we cannot construct the compact absorbing set in this topology. Moreover, the elementary examples show that problem (1.1) really may not possess the attractor in a 'uniform' topology.)

Thus, the precompact absorbing set $\mathbb{V}_R$ is already constructed and it remains to find the compact one. The most simple way is to take the compact absorbing set $\mathbb{V}'_R := [\mathbb{V}_R]_\Phi$, where $[\cdot]_\Phi$ means the closure in Φ. Indeed, since $\mathbb{V}_R \subset \mathbb{M}_R \subset K^+$ and $\mathbb{M}_R$ is evidently closed in Φ then $\mathbb{V}'_R \subset K^+$ and consequently it is really the compact absorbing set for semigroup (3.2). Thus (due to the attractor's existence theorem for abstract semigroups) semigroup (3.2) possesses an attractor $\mathcal{A}_{tr}$ which can be defined by formula

$$\mathcal{A}_{tr} = \cap_{h\geq 0}\left[\cup_{s\geq h} T_s\mathbb{V}'_R\right]_\Phi \tag{3.12}$$

As usual representation (3.6) is a standard corollary of definition (3.12) (see [1, 24]) but since this representation is of fundamental significance for our purposes we recall shortly its proof. Indeed, let $\widehat{u}(x)$, $x \in \Omega$ be a non-negative bounded solution of problem (3.7). Then, particularly the sequence $\Pi_{\Omega_+}(T_{-h}\widehat{u})$, $h \in \mathbb{N}$ is uniformly bounded in $C_b^{2+\alpha}(\Omega_+)$, consequently, according to the attractor's definition

$$T_h\Pi_{\Omega_+}(T_{-h}\widehat{u}) \to \mathcal{A}_{tr} \text{ in } \Phi \text{ as } h \to \infty$$

From the other side $T_h\Pi_{\Omega_+}(T_{-h}\widehat{u}) = \Pi_{\Omega_+}\widehat{u}$. Thus, $\Pi_{\Omega_+}\widehat{u} \in \mathcal{A}_{tr}$ and consequently

$$\Pi_{\Omega_+}K(\Omega) \subset \mathcal{A}^{tr} \tag{3.13}$$

Let us prove the opposite including. Let $u \in \mathcal{A}_{tr}$. Then (3.12) implies that there are a sequence $h_n \to +\infty$ and a sequence of solutions $u_n \in \mathbb{V}'_R$ such that

$$u = \Phi\text{--}\lim_{n\to\infty} T_{h_n} u_n \tag{3.14}$$

Note that the solution $T_{h_n}u_n$ is defined not only in Ω_+ but in the domain $T_{h_n}\Omega_+ := (-h_n, \infty) \times \Omega_0$ and

$$\|u_n\|_{C_b^{2+\beta}(T_{h_n}\Omega_+)} \le R \tag{3.15}$$

And consequently arguing as in the proof of compactness of $\mathbb{V}'_R$ we derive that the sequence $T_{h_n}u_n$, $n \ge n_0$ is precompact in $C^{2+\beta}_{loc}(T_{h_{n_0}+1}\Omega_+)$ for every $n_0 \in \mathbb{N}$. Passing to a subsequence if necessary and using the Cantor diagonal procedure and the fact that $h_n \to \infty$ we may assume that this sequence converges to a some function $\widehat{u} \in C^{2+\beta}_{loc}(\overline{\Omega})$ in the spaces $C^{2+\beta}_{loc}(T_{h_{n_0}+1}\Omega_+)$ for every $n_0 \in \mathbb{N}$. Then (3.15) implies that $\widehat{u} \in C_b^{2+\beta}(\Omega)$. Moreover since $T_{h_n}u_n$ are the non-negative solutions of (1.1) then passing to the $n \to \infty$ we easily obtain that $\widehat{u}$ is a non-negative solution of equation (3.7) and formula (3.14) gives us that $\Pi_{\Omega_+}\widehat{u} = u$. Thus, $u \in \Pi_{\Omega_+}K(\Omega)$. Theorem 3.1 is proved.

REMARK 3.1. *Note that neither our concrete choice of of the domain $\Omega_+ = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n$ nor the concrete choice of the 'time' direction x_1 are not essential for the of the trajectory dynamical system approach. Indeed, let us replace the 'time' direction x_1 by any fixed direction $\vec{l} \in \mathbb{R}^{n+2}$ and (and correspondingly $(T_h u)(x) := u(x + h\vec{l})$. Then the above construction seems to be applicable if the domain Ω_+ satisfies the following assumptions:*

1. $T_h\Omega_+ \subset \Omega_+$ (it is necessary in order to define the restriction T_h to the trajectory phase space K_+).

2. $\Omega = \cup_{h\le 0} T_{-h}\Omega_+$ (it is required in order to obtain representation (3.6)).

The trajectory attractor for the elliptic problems in the domains Ω_+ which satisfy 1 and 2 with $\Omega = \mathbb{R}^{n+2}$ has been constructed in [27]. The result of Theorem 3.1 shows that this approach works not only for Ω_+ such that in the condition 2 $\Omega = \mathbb{R}^{n+2}$ but for a larger class of domains Ω_+.

4. Symmetry and stabilization.

In this Section we restrict ourselves to consider only the where Ω_+ is 3-dimensional $(n = 1)$. Then, using the description for the positive bounded solutions of (3.7) given in [9], we obtain the additional information about the behavior of solutions of initial problem (1.1).

PROPOSITION 4.1. *Let assumptions (1.2) hold and let $n = 1$. Then any non-negative bounded solution $\widehat{u}(x)$ of equation (3.7) depends only on the variable x_2, i.e $u(x) = V(x_2)$ where $V(z)$ is a bounded solution of the following problem:*

$$V''(z) - f(V(z)) = 0, \quad z > 0, \quad V(0) = 0, \quad V(z) \ge 0 \tag{4.1}$$

The proof of this Proposition is given in [9] for the case where the solution $\widehat{u}(x)$ is strictly positive inside of Ω. The general case can be reduced to this one using the following version of the strong maximum principle.

LEMMA 4.1. ([9]) *Let $V \subset \mathbb{R}^n$ be a (connected) domain with the sufficiently smooth boundary and let $w \in C^2(V) \cap C(\overline{V})$ satisfy the following inequalities*

$$\Delta_x w(x) - l(x)w(x) \le 0, \quad x \in V, \quad w(x) \ge 0, \quad x \in V \tag{4.2}$$

Assume also that $|l(x)| \le K$ for $x \in V$. Then either $v(x) \equiv 0$ or $v(x) > 0$ for every interior point $x \in V$.

In order to apply the lemma to equation (3.7) we rewrite it in the following form

$$\Delta_x \widehat{u} - l(x)\widehat{u} = f(0) \le 0, \ \ l(x) := \frac{f(\widehat{u}(x)) - f(0)}{\widehat{u}(x)} \tag{4.3}$$

Since $f \in C^1$ and the solution $\widehat{u}(x)$ is bounded then $l(x)$ is also bounded in Ω. Thus, according to Lemma 1 either $\widehat{u}(x) \equiv 0$ (which is evidently symmetric) or $\widehat{u}(x) > 0$ in the interior of Ω and then Proposition 4.1 follows from the result of [9] mentioned above. Proposition 4.1 is proved.

Denote by $\mathcal{R}_V$ the set of all bounded non-negative solutions $V(z)$ of problem (4.1). Then Proposition 4.1 implies that

$$\mathcal{A}_{tr} = \mathcal{R}_V \tag{4.4}$$

Let us study now the positive solutions of problem (4.1). It is well known that every non-negative bounded solution of this problem should be monotonically increasing $V(z_1) \ge V(z_2)$ if $z_1 \ge z_2$, consequently there is a limit

$$z_0 = z_0(V) := \lim_{z \to +\infty} V(z), \ f(z_0) = 0, \ \ 0 \le V(z) \le z_0, \ \ z \ge 0 \tag{4.5}$$

Moreover, it follows from Lemma 4.1 that either $V(z) \equiv 0$ or $V'(z) > 0$ for every $z \ge 0$.

Multiplying equation (4.1) by V' and integrating over $[0, z]$ we obtain the explicit expression for the derivative $V(z)$

$$V'(z)^2 = -2F(V(z)) + C \tag{4.6}$$

where $F(V) := -\int_0^V f(V)\, dV$.

Passing to the limit $z \to +\infty$ in (4.6) and taking into account (4.5) one can easily derive that $C = 2F(z_0)$. Therefore we obtain the following equation for $V(z)$, stabilizing to z_0:

$$V'(z)^2 = 2(F(z_0) - F(V(z)) \tag{4.7}$$

Assume now that $F(z_0) \ge 0$ (in the other case $V(z) \equiv 0$). Then the solution $V_{z_0}(z)$ of (4.7) which satisfies (4.5) exists if and only if $F(z_0) - F(z) > 0$ for every $z \in (0, z_0)$. Moreover, such solution is unique because V_{z_0} satisfies (4.1) with the initial conditions

$$V_{z_0}(0) = 0, \ \ V'_{z_0}(0) = \sqrt{2F(z_0)} \tag{4.8}$$

Denote

$$\mathcal{R}_f^+ := \{z_0 \in \mathbb{R}_+ : f(z_0) = 0, \ \ F(z_0) - F(z) > 0 \text{ for every } z \in (0, z_0)\} \tag{4.9}$$

Note that set (4.9) is totally disconnected in $\mathbb{R}$. Indeed, otherwise it should contain a segment $[\alpha, \beta] \in \mathcal{R}_f^+$, $\beta > \alpha \ge 0$. Then, $f(z_0) \equiv 0$ for $z_0 \in [\alpha, \beta]$ and consequently $F(z_0) = F(\beta)$ for every $z_0 \in [\alpha, \beta]$, which evidently contradicts the fact that $\beta \in \mathcal{R}_f^+$.

Thus, we obtain the following proposition.

Proposition 4.2. *There is a homeomorphism*

$$\tau : (\mathcal{R}_V, C^{2+\beta}_{loc}(\mathbb{R}_+)) \to (\mathcal{R}^+_f, \mathbb{R})$$

Moreover, the set $\mathcal{R}^+_f$ and (consequently) $\mathcal{R}_V$ are totally disconnected.

Indeed, (4.8) defines a homeomorphism between $\mathcal{R}^+_f$ and the set $\mathcal{R}_V(0) := \{(0, V'(0)) : V \in \mathcal{R}_V\}$ of values at $t = 0$ for functions from $\mathcal{R}_V$. Recall that $\mathcal{R}_V$ consists of solutions of second order ODE (4.1) and, consequently, thanks to a classical theorem on continuous dependence of solutions of ODE's, the set $\mathcal{R}_V$ is homeomorphic to $\mathcal{R}_V(0)$ and this homeomorphism is given by the solving operator $S : (V(0), V'(0)) \to V(t)$ of equation (4.1). Proposition 4.2 is proved.

Remark 4.1. *Note that, although for generic fs the sets $\mathcal{R}^+_f \sim \mathcal{R}_V$ are finite, these sets maybe even uncountable for some very special choices of the nonlinearity f. The simplest example of such f is the following one:*

$$f(z) = -\operatorname{dist}(z, K) \tag{4.10}$$

where K is a standard Cantor set on $[0,1]$ and $\operatorname{dist}(z,K)$ means a distance from z to K. Indeed, it is easy to verify that for this case $\mathcal{R}^+ = K$ and consequently $\mathcal{R}_V$ consists of continuum elements. (To be rigorous, function (4.10) is only Lipschitz continuous (but not from C^1) and does not satisfy also the second assumption of (1.2), but slightly modifying this function one can construct the function $\tilde{f}$, which will satisfy all our assumptions and $\mathcal{R}^+_{\tilde{f}} = \mathcal{R}^+_f = K$.)

We state now the main result of this Section.

Theorem 4.3. *Let the assumptions of Proposition 4.1 hold. Then for every nonnegative bounded solution u of problem (1.1) there is a solution $V(x_2) = V_u(x_2) \in \mathcal{R}_V$ of problem (4.1) such that for every fixed R and $x = (x_1, x_2, x_3)$*

$$\|u - V_u\|_{C^{2+\beta}(B^R_{x_h} \cap \Omega_+)} \to 0, \ \ x_h := (x_1 + h, x_2, x_3) \tag{4.11}$$

when $h \to \infty$.

Proof. Indeed, consider the ω-limit set of the solution $u \in K^+$ under the action of the semigroup T_h of shift in the x_1 direction:

$$\omega(u) = \cap_{h \geq 0} \big[\cup_{s \geq h} T_s u \big]_\Phi \tag{4.12}$$

Recall that T_h possesses the attractor $\mathcal{A}_{tr}$ in K^+, consequently, set (4.12) is non-empty and

$$\omega(u) \subset \mathcal{A}_{tr} \tag{4.13}$$

It follows now from Proposition 4.1 that $\omega(u) \subset \mathcal{R}_V$.

Note that from the one side the set $\omega(u)$ must be connected (see e.g. [15]) and from the other side it is a subset of the set $\mathcal{R}_V$ which is totally disconnected (due to Proposition 4.2). Therefore $\omega(u)$ consists of a single point $V_u \subset \mathcal{R}_V$:

$$\omega(u) = \{V_u\} \tag{4.14}$$

The assertion of the theorem is a simple corollary of this fact and of our definition of the topology in K^+. Theorem 4.1 is proved.

5. Concluding remarks.

In conclusion of the paper we discuss assumptions (1.2) imposed to the nonlinear term $f(u)$ in order to obtain the results of Section 4. Note firstly that sign condition (1.2)(3) is evidently essential in order to prove the solvability of (1.1) in the class of positive bounded solutions for every positive bounded initial data u_0 (and in a fact it is also essential for Proposition 4.1 and Lemma 4.1, see e.g. [9]).

The assumption $f \in C^1$ is not necessary neither for proving the existence of a positive solution of problem (1.1) (in Section 2) nor for applying the trajectory dynamical system approach to this problem (see Section 3) and can be weakened to $f \in C(\mathbb{R}, \mathbb{R})$, see [25] or [27]. Note however that the (local) Lipschitz continuity of the nonlinear term is very essential for the symmetry result, formulated in Proposition 4.1 (see [9]) and consequently for all results, obtained in Section 4.

Thus, assumptions (1.2)(1) and (1.2)(3) seem to be close to optimal in order to derive the results of Section 4. In contrast to them, 'dissipativity assumption' (1.2)(2) is far from optimal and has been imposed in such a form only in order to avoid the additional technicalities and to make the trajectory approach to study the behavior of positive solutions more clear. In a fact, it can be proven using the standard sub and supersolutions technique and some monotonicity results for positive solutions of elliptic equations that under assumptions (1.2)(1) and (1.2)(3) problem (1.1) possesses at least one positive bounded solution for every positive bounded initial value u_0 if and only if it's one dimensional analogue

$$V''(z) - f(V(z)) = 0, \quad z > 0, \quad V(0) = M \tag{5.1}$$

is solvable in the class of bounded nonnegative solutions for every $M \geq 0$. Remind that (5.1) is second order ODE of Newtonian type and can be easily analyzed using e.g. the phase portrait technique.

In the case where $n = 1$ using the explicit description of the set of bounded positive solutions of the equation (1.1) in Ω one can easily show that the attractor $\mathcal{A}_{tr}$ exists if and only if the set K of all bounded positive solutions of the problem

$$V''(z) - f(V(z) = 0, \; z > 0, \; V(0) = 0 \tag{5.2}$$

is globally bounded in $C(\mathbb{R}_+)$.

Combining (5.1) and (5.2) we derive after the straightforward analysis of the corresponding phase portrait that under assumptions (1.2)(1) and (1.2)(3), problem (1.1) possesses the trajectory attractor $\mathcal{A}_{tr}$ if and only if the potential $F(v) := -\int_0^v f(u)\, du$ achieves itrs global maximum on $[0, \infty)$, i.e. if there is $v_0 \geq 0$ such that

$$F(z_0) = \max_{v \in \mathbb{R}_+} F(v) \tag{5.3}$$

Hence, all results of Section 4 remain valid in the case where condition (1.2)(2) is replaced by (5.3). Evidently, condition (1.2)(2) is sufficient, but not necessary for (5.3).

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